A STATE-SPACE ALGORITHM FOR THE SUPEROPTIMAL HANKEL-NORM APPROXIMATION PROBLEM*

G. D. HALIKIAS†, D. J. N. LIMEBEER‡, AND K. GLOVER§

Abstract. It has been demonstrated by N. T. Young [NATO ASI Series F34, Springer-Verlag, Berlin, New York, 1987] that given a stable matrix-valued function $G_0(s)$ and a nonnegative integer $k$, there exists a unique superoptimal approximation $\Phi(s)$ with no more than $k$ poles in the left half plane that minimizes the sequence $(s_1^*(G_0 + \Phi), s_2^*(G_0 + \Phi), \ldots )$, with respect to lexicographic ordering, where $s_i^*(G_0 + \Phi) := \sup_{\omega} |s_i(G_0 + \Phi)(j\omega)|$ and $s_i(\cdot)$ are the singular values in descending order of magnitude. This paper presents a constructive state-space algorithm that evaluates the superoptimal approximating matrix function. The procedure recursively minimizes each frequency-dependent singular value with the aid of all-pass transformations constructed from the $k$th Schmidt pairs of a sequence of Hankel operators. The algorithm may be stopped after an arbitrary number of, say, $l \leq \min(m, p)$ steps. The representation formula at the $l$th stage will characterize all matrix functions that have $\leq k$ poles in the left half plane and that minimize $s_i^*(G_0 + \Phi), \ldots, s_l^*(G_0 + \Phi)$.

Key words. model reduction, superoptimality, Hankel norm, Schmidt vectors

AMS subject classifications. 93B28, 93B40, 93B36

1. Introduction. There are many occasions on which engineers require reliable low-order approximations to high-order models. For this reason the model-order-reduction problem has been the subject of numerous theoretical investigations, and several different approaches have been developed. A technique that has received much recent attention is the optimal Hankel-norm approach [4], which offers good guaranteed performance characteristics that are close to verifiable lower bounds.

For matrix-valued problems the optimal Hankel-norm approach typically has a continuum of solutions. The question then arises as to which solution (if there is one) is best. A partial answer to this question is implicit in [4], in that $L^\infty$-error bounds are available for only certain optimal Hankel-norm reduced-order models. Young discusses an alternative approach to the uniqueness question [16]. His suggestion is to seek to minimize the sequence $s^*(E) = (s_1^*(E), s_2^*(E), \ldots )$ rather than just $s_i^*(E)$, where $s_i^*(E) := \sup_{\omega} |s_i(E)(j\omega)|$, $E(s)$ is the modeling error, and $s_i(\cdot)$ is the $i$th singular value (numbered in descending order of magnitude). The reduced-order model that minimizes $s^*(\times E)$ has been shown to exist and to be unique [16]. In model-reduction applications it is possible to reduce the $L^\infty$-norm of the error system by using the superoptimal Hankel-norm approximation rather than some other approximation, but we have no proof of this. For diagonal problems the superoptimal solution is the most natural choice because it is the diagonal matrix of optimal solutions.

The idea behind superoptimality is easily illustrated by way of a simple $2 \times 2$ example. Suppose

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} = \text{diag} \{ g_1(s), g_2(s) \},$$

$$\text{(1.1)}$$

* Received by the editors January 1, 1991; accepted for publication (in revised form) January 30, 1992.
† Department of Electrical Engineering, University of Leeds, Leeds, United Kingdom.
‡ Department of Electrical Engineering, Imperial College, Exhibition Road, London, United Kingdom.
§ Department of Engineering, Trumpington Street, Cambridge, United Kingdom.
in which \( \sigma_i(G) = 1/2 \). Suppose also that a reduced-order approximation in \( \mathcal{H}_\infty^c(1) \) is required. Any solution of the form

\[
F(s) = \text{diag} \{ f_1(s), -\frac{1}{2} \}
\]

is an optimal approximation to \( G(s) \), provided that \( f_1(s) \) is chosen to be in \( \mathcal{H}_\infty^c(1) \) such that

\[
\left\| \frac{2}{s+1} + f_1 \right\|_{\infty} \leq \frac{1}{2}.
\]

For example, the solution

\[
F_{ap}(s) := \text{diag} \left\{ -\frac{s+3}{2(s+1)}, \frac{1}{2} \right\}
\]

results in the all-pass error system

\[
(G + F_{ap})(s) = \begin{bmatrix}
-\frac{1}{2} \left( \frac{s-1}{s+1} \right) & 0 \\
0 & -\frac{1}{2} \left( \frac{s-1}{s+1} \right)
\end{bmatrix},
\]

and in this case \( (s_1^\infty(G + F_{ap}), s_2^\infty(G + F_{ap})) = (\frac{1}{2}, \frac{1}{2}) \). It is clear, however, that we can do better than this. Suppose we use our one stable pole to cancel \( g_1(s) \). This gives

\[
F_{so}(s) = \begin{bmatrix}
-\frac{2}{s+1} & 0 \\
0 & -\frac{1}{2}
\end{bmatrix},
\]

resulting in an error system

\[
(G + F_{so})(s) = \begin{bmatrix}
0 & 0 \\
0 & -\frac{1}{2} \left( \frac{s-1}{s+1} \right)
\end{bmatrix},
\]

for which \( (s_1^\infty(G + F_{so}), s_2^\infty(G + F_{so})) = (\frac{1}{2}, 0) \). This is the superoptimal approximation to \( G(s) \). If the \((1, 2)\) and \((2, 1)\) elements of \( G(s) \) are nonzero, the situation is more complicated and a formal algorithmic procedure is required.

This paper recasts Young's algorithm in a concrete state-space framework that can be implemented on a digital computer that can tackle any rotational superoptimal approximation problem. Section 2 contains the notation to be used and a standard Hankel-norm approximation result. Section 3 contains the main results of the paper: Theorem 3.1 is standard and describes a key property of the Schmidt pairs of Hankel operators. Lemma 3.2 is a modified version of a result in [9], and Lemmas 3.3 and 3.4 are generalizations of parallel results in [9]. The main results of the paper are Theorems 3.6 and 3.6', and Algorithm 3.1, which are believed to be new. The main conclusions of our work are given in § 4.
2. Notation and preliminaries.

2.1. Notation.

- \( \mathbb{R}, \mathbb{R}_+, \mathbb{C} \): real, nonnegative, and complex numbers
- \( \mathbb{R}(s) \): field of rational functions in \( s \) with real coefficients
- \( \mathbb{C}_+, \mathbb{C}_- \): open (respectively, closed) right half plane
- \( \mathbb{C}_{-0}, \mathbb{C}_{-0} \): open (respectively, closed) left half plane
- \( \lambda(A), \lambda_{\text{max}}(A) \): spectrum of a square matrix \( A \), largest eigenvalue of \( A \)
- \( A^* \): complex conjugate transpose of \( A \) \( \subseteq \mathbb{C}^{p \times m} \)
- \( A \geq 0, A > 0 \): \( A \) is positive semidefinite (respectively, positive definite)
- \( A \leq 0, A < 0 \): \( A \) is negative semidefinite (respectively, negative definite)
- \( A^\approx \): generalized inverse of matrix \( A \)
- \( \mathcal{L}^\infty(p \times m) \): space of \( p \times m \) matrix functions with entries that are bounded on the \( j\omega \) axis (including the point at \( \infty \))
- \( H^\infty_+, H^\infty_- \): \( H^\infty \) norm of matrices in \( \mathcal{L}^\infty \)
- \( \mathcal{H}^\infty_+, \mathcal{H}^\infty_- \): subspace of \( \mathcal{L}^\infty \); \( p \times m \) matrix functions that are analytic and bounded in \( \mathbb{C}_+ \) (respectively, \( \mathbb{C}_- \))
- \( \mathcal{H}^\infty_2, \mathcal{H}^\infty_- \): the sets of functions \( f \) analytic in \( \mathbb{C}_+ \) (respectively, \( \mathbb{C}_- \)) such that
- \( \mathcal{H}^\infty_2 (p \times m)(s) \): the same as \( \mathcal{H}^\infty_2 \) except that elements are taken from \( \mathbb{R}(p \times m)(s) \)
- \( \mathcal{H}^\infty_+ (p \times m)(k) \): the set of \( p \times m \) matrix functions in \( \mathcal{L}^\infty \) with no more than \( k \) poles in \( \mathbb{C}_+ \)
- \( \mathcal{B} \mathcal{H}^\infty_+, \mathcal{B} \mathcal{H}^\infty_- \): unit balls in \( \mathcal{H}^\infty_+ \); \{ \( f \in \mathcal{H}^\infty_+ \) : \( \| f \|_\infty \leq 1 \) \}, \{ \( f \in \mathcal{H}^\infty_- \) : \( \| f \|_\infty \leq 1 \) \}
- \( \mathcal{H}^\infty_2 (p \times m)(s) \): the same as \( \mathcal{H}^\infty_2 (p \times m)(s) \) except that elements are taken from \( \mathbb{R}(p \times m)(s) \)
- \( \mathcal{H}^\infty_+ (p \times m)(k) \): the same as \( \mathcal{H}^\infty_+ (p \times m)(k) \) except that elements are taken from \( \mathbb{R}(p \times m)(s) \)
- \( \Gamma_G \): Hankel operator with symbol \( G(s) \in \mathcal{H}^\infty_+ \)
- \( \sigma_i(G(s)) \): \( i \)th Hankel singular value of \( G(s) \) (i.e., of \( \Gamma_G \)) in descending order of magnitude
- \( [F]_+ \): the stable projection of \( F \) if \( F \) is decomposed as \( F := [F]_+ + [F]_- \) in which \( [F]_+ \in \mathcal{H}^\infty_+ \) and \( [F]_- \in \mathcal{H}^\infty_- \)
- \( s_i(A) \): \( i \)th singular value of a matrix \( A \) with the numbering in descending order (if \( A \) is a function of frequency (i.e., \( A(j\omega) \)) then \( s_i(\cdot) \) will be a function of frequency also)
- \( \Pi_{i=0}^n A_i \): (right) matrix product \( A_0A_1 \cdots A_n \) (\( \Pi_{i=0}^n A_i := A_0 \))
- \( \| G(s) \|_H \): the Hankel-norm of \( G(s) \)
- \( G^*(s) \): \( G(-s)^* \), the para-Hermitian conjugate of \( G(s) \)
- \( C(A, B), \tilde{C}(A, B) \): controllable and uncontrollable modes of the pair \( (A, B) \)
- \( O(A, C), \tilde{O}(A, C) \): observable and unobservable modes of the pair \( (A, C) \)
- \( A \oplus B \): direct sum given by

\[
\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix}
\]
Associated with a transfer function matrix $G_0(s) \in \mathbb{R}(s)^{n \times m}$ of MacMillan degree $n$ is a state-space realization

$$G_0(s) = D + C(sI - A)^{-1}B,$$  \hspace{1cm} (2.1)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$. We will use the alternative notation $G_0(s) \equiv (A, B, C, D)$ or

$$G_0(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$  \hspace{1cm} (2.2)

for realizations of $G_0(s)$. The rank of $G_0(s)$ is taken to be its rank for any $s$ that is not a zero of $G_0(s)$.

In the above notation we have $G_0^*(s) \equiv (-A^*, C^*, -B^*, D^*)$, and if $D$ is nonsingular, we have $G_0^{-1}(s) \equiv (A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$. If $G^{-1}(s) = G^*(s)$, then $G(s)$ is called all-pass. $G_0(s)$ is called stable if it has no poles in $\overline{C}_+$. If $G_0(s)$ is both stable and all-pass, it is called inner.

We will talk about basis changes $T$ in the state space of $G_0(s)$; we will take a basis change to mean $G_0(s) \equiv (A, B, C, D) \overset{T}{\rightarrow} G_0(s) = (TAT, TB, CT, D)$. The MacMillan degree of $G_0(s)$ will be written $\text{deg}(G_0)$, and the set of poles (zeros) of $G_0(s)$ will be denoted $\{\text{poles of } G_0\}$ (\{zeros of $G_0\}$).

Let $P(s)$ be a partitioned matrix with a state-space realization given by

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix},$$  \hspace{1cm} (2.3)

Then

$$P_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij}$$  \hspace{1cm} (2.4)

is a state-space realization of $P_0(s)$. A linear fractional transformation for the partitioned matrix $P$ and a matrix $K$ is defined as

$$\mathcal{F}_i(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21},$$  \hspace{1cm} (2.5)

where $K$ is of dimension $l \times m$ if $P_{22}$ has dimension $m \times l$.

### 2.2. Preliminaries

This section provides a description of all $k$th-order approximations of a rational transfer function $G_0(s) \in \mathbb{R}(s)^{n \times m}$. The description is in terms of a balanced realization of $G_0(s)$ and is based on [4, Thm. 8.7]. If $G_0(s) \equiv (A, B, C, D)$ is balanced and minimal, the following Lyapunov equations are satisfied:

$$AP + PA^* + BB^* = 0,$$  \hspace{1cm} (2.6)

$$A^*Q + QA + CC^* = 0,$$  \hspace{1cm} (2.7)

in which

$$P = Q = \text{diag} (\Sigma, \sigma_{k+1}),$$  \hspace{1cm} (2.8)

where

$$\Sigma = \text{diag} (\sigma_1, \ldots, \sigma_k, \sigma_{k+2}, \ldots, \sigma_n).$$  \hspace{1cm} (2.9)

**Remark 2.1.** In the interests of a clear presentation, it is assumed that the $(k + 1)$th Hankel singular value is nonrepeated. Matrices $A$, $B$, and $C$ are partitioned conformally with $P$ and $Q$ in (2.8) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ C_2].$$  \hspace{1cm} (2.10)
In a further attempt to keep the notation simple, it will be assumed that $G_0(s)$ has been scaled to give $C_2^* C_2 = B_2^* B_2 = 1$.

Lemma 2.1 parameterizes the family of all optimal Hankel-norm approximations and their corresponding extensions in terms of an (arbitrary) unstable contraction [4].

**Lemma 2.1.** Let the transfer function $G_0(s) \in \mathbb{R}^{p \times m}$ have a stable, minimal, and balanced realisation $G_0(s) = (A, B, C, D)$ with Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_k > \sigma_{k+1} \geq \cdots \geq \sigma_n > 0$, and define

$$\Gamma = \Sigma^2 - \sigma_{k+1}^2 I_{n-1}. \tag{2.11}$$

Then all error systems $\mathcal{E}(s) = G_0(s) + \mathcal{F}(s)$ with

$$\|\mathcal{E}(s)\|_\infty = \sigma_{k+1} := \sigma \tag{2.12}$$

and $\mathcal{F}(s) \in \mathcal{H}^\infty(k)$ are generated by

$$\mathcal{E}(s) = \mathcal{F}\left( H(s), \frac{1}{\sigma} \Theta(s) \right), \quad \Theta(s) \in \mathcal{BH}^\infty, \tag{2.13}$$

in which $H(s)H^*(s) = \sigma^2 I$ and

$$H(s) = \begin{bmatrix} G_0 + F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{2.14}$$

where

$$F(s) = \begin{bmatrix} \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} + \sigma C_{1}^* UB_{1}^*) & \Gamma^{-1}(\Sigma B_{1} + \sigma C_{1}^* U) & -\sigma \Gamma^{-1} C_{1}^* C_{2}^* \\ -\sigma _{k+1}^2 B_{2} & \sigma U & -\sigma C_{2}^* \\ -\sigma _{k+1}^2 C_{2} & \sigma B_{2} & 0 \end{bmatrix}. \tag{2.15}$$

Also, $F(s) \in \mathcal{R}^{\infty}(p \times m - 1)(k)$. The matrices $B_{2}$ and $C_{2}$ are chosen to make $[C_2, C_2^*]$ and $[B_2^*, B_2^*]$ orthogonal, and $U$ is given by $U := -C_2 B_2$. In the single-input or single-output case $B_{2}$ or $C_{2}$ or both will be zero and the resulting error system will be unique.

**Remark 2.2.** It is interesting that the construction of the superoptimal approximation is particularly simple in the special case $k = n - 1$, for which the approximation can be obtained directly from Lemma 2.1. To see this we begin by noting that all error systems are given by

$$\mathcal{E}(s) = G + F_{11} + \frac{1}{\sigma_n} F_{12} \Theta \left( I - \frac{1}{\sigma_n} F_{22} \Theta \right)^{-1} F_{21}, \tag{2.16}$$

in which $F(s)$ is stable. Suppose that we set $\Theta := (1/\sigma_n) F_{22}^*(s)$, which we note has the characteristic $\|\Theta\|_\infty < 1$ by the all-pass property $H(s)H^*(s) = \sigma_n^2 I$ and since det $(F_{22}^*, F_{12}(j\omega)) \neq 0$ for all real $\omega$. In addition, $F_{22}^*$ is completely unstable. By exploiting the all-pass character of (2.14), the expression for the error system may be rearranged as

$$\mathcal{E}(s) = G + F_{11} + \frac{1}{\sigma_n} F_{12} F_{22}^* \left( I - \frac{1}{\sigma_n} F_{22} F_{22}^* \right)^{-1} F_{21}, \tag{2.17}$$

$$= G + F_{11} + F_{12} F_{22}^* (F_{22} F_{22}^*)^{-1} F_{21}$$

$$= G + F_{11} - (G + F_{11}) F_{22}^* (F_{22} F_{22}^*)^{-1} F_{21}$$

$$= (G + F_{11})(I_m - F_{22}^* (F_{22} F_{22}^*)^{-1} F_{21}).$$
Since $F_{21}(j\omega)$ has rank $m-1$, the error system has rank 1 and satisfies $\|\mathcal{E}\|_\infty = \sigma_m$ which is clearly superoptimal.

To show that the error system in (2.17) is unique, we suppose there are two superoptimal approximations $\Phi_{so1}$ and $\Phi_{so2}$. It follows from Lemma 3.4 and the unity-rank property of $\mathcal{E}(s)$ in (2.17) that there exists an all-pass matrix $W(s) = [w(s) \ W_{\perp}(s)]$ such that
\[
(G + \Phi_{so1}) W(s) = [\sigma, a(s) v(s) \ 0],
\]
\[
(G + \Phi_{so2}) W(s) = [\sigma, a(s) v(s) \ 0].
\]
Subtracting these expressions gives
\[
(\Phi_{so1} - \Phi_{so2}) W(s) = 0 \Rightarrow \Phi_{so1} - \Phi_{so2}.
\]

3. Main results. In this section we present the main algorithm for calculating superoptimal approximations. Following the work of Young [15], [16], the procedure is based on an inductive dimension-peeling argument. At each step of the algorithm the rank of the problem is reduced by one. Since the original problem is assumed to be of finite rank, the algorithm terminates after a finite number of steps.

The Hankel operator induced by $G_0(s)$ is defined by $F_{G_0} = Y^{-1} G_0 Y$, where $I + MG_0$ denotes the projection $L_2 \rightarrow \mathcal{H}_2$ and $M_{G_0}$ is the multiplication operator. Note that $F_{G_0}$ is determined by the stable component of $G_0(s)$. We begin our development by briefly mentioning some elementary properties of the Schmidt vectors of $\Gamma_{G_0}$. The interested reader is referred to [1], [13] for a more detailed exposition. Suppose that $\sigma_{k+1}$ is a singular value of $\Gamma_{G_0}$. Then there exist Schmidt vectors $f_{k+1}(s) \in \mathcal{H}_2$ and $g_{k+1}(s) \in \mathcal{H}_2^\perp$ that satisfy
\[
\Gamma_{G_0} g_{k+1} = \sigma_{k+1} f_{k+1}
\]
and
\[
\Gamma_{G_0}^* f_{k+1} = \sigma_{k+1} g_{k+1},
\]
and consequently
\[
\Gamma_{G_0}^* \Gamma_{G_0} g_{k+1} = \sigma_{k+1}^2 g_{k+1}.
\]

The next result is standard (see, e.g., [13]) and demonstrates that any Schmidt pair of $\Gamma_{G_0}$ has singular-vector-type properties for the error system $(G_0 + F)$. If $F(s) \in \mathcal{H}_\infty(k)$ is any optimal approximation of $G_0(s) \in \mathcal{H}_\infty$, then
\[
(G_0 + F) g_{k+1} (s) = \sigma_{k+1} f_{k+1} (s), \quad (G_0 + F)^* f_{k+1} (s) = \sigma_{k+1} g_{k+1} (s).
\]
Thus by modulo scaling $f_{k+1}(s)$ and $g_{k+1}(s)$ are singular vectors of the error system $E(s) = (G_0 + F)(s)$ at each frequency $s = j\omega$ corresponding to the largest singular value of $E(j\omega)$.

Theorem 3.1. Every $F(s) \in \mathcal{H}_\infty(k)$ that achieves the infimum
\[
\inf_{F(s) \in \mathcal{H}_\infty(k)}\|G_0 + F\|_\infty = \sigma_{k+1}(G_0) < \sigma_k(G_0)
\]
satisfies
\[
(G_0 + F) g_{k+1} (s) = \Gamma_{G_0} g_{k+1} (s) = (G_0 + F) f_{k+1} (s),
\]
where $(g_{k+1}, f_{k+1})$ is a Schmidt pair of $\Gamma_{G_0}$ corresponding to the $(k+1)$th singular value of $\Gamma_{G_0}$.

Proof. Let $F(s)$ be any (matrix) function that achieves the infimum in (3.5), and let $F$ be the Hankel operator that it induces. Since $F(s) \in \mathcal{H}_\infty$, rank $(\Gamma_F) \leq k$. 0321"03019
Suppose \((g_i, f_i), (i = 1, 2, \ldots),\) are the Schmidt pairs of \(\Gamma \) associated with \(\alpha_i\), and define \(P\) to be the orthogonal projection onto \(\text{Span} (f_1, f_2, \ldots, f_{k+1})\). Then \(\|\Gamma \| = \sigma_{k+1}(\Gamma)\) implies that

\[
\| P(\Gamma \Gamma) \| \leq \sigma_{k+1}(\Gamma) .
\]

The operator \(P(\Gamma)\) mapping \(\text{Span} (g_1, g_2, \ldots, g_{k+1}) \rightarrow \text{Span} (f_1, f_2, \ldots, f_{k+1})\) has rank at most \(k\), and therefore there exists a function of norm equal to one such that

\[
x = \sum_{i=1}^{k+1} \alpha_i g_i \in \text{Ker} (P(\Gamma)).
\]

Consequently,

\[
\| P(\Gamma \Gamma)x \|_2 = \| P(\Gamma) \|_2 \leq \sigma_{k+1}(\Gamma).
\]

Also,

\[
\| P(\Gamma \Gamma) \left( \sum_{i=1}^{k+1} \alpha_i g_i \right) \|_2 = \| P \left( \sum_{i=1}^{k+1} \alpha_i \Gamma g_i \right) \|_2 = \| P \left( \sum_{i=1}^{k+1} \sigma_i \alpha_i f_i \right) \|_2
\]

\[
= \| \sum_{i=1}^{k+1} \sigma_i \alpha_i f_i \|_2
\]

\[
= \sqrt{\sum_{i=1}^{k+1} |\alpha_i|^2}
\]

since the \(f_i\)’s are orthonormal. This implies that \(\sum_{i=1}^{k+1} |\alpha_i|^2 \leq \sigma_{k+1}^2\), and since \(\sigma_{k+1} < \sigma_k < \cdots < \sigma_1\) and \(\sum_{i=1}^{k+1} |\alpha_i|^2 = 1\), we conclude that \(\alpha_i = 0\) for \(i = 1, 2, \ldots, k\), so that \(x\) must be a multiple of \(g_{k+1}\), say, \(x = \beta g_{k+1}\) with \(|\beta| = 1\). Now, since \(x \in \text{Ker} (P(\Gamma))\), we have that \(\Gamma x \perp \text{Span} (f_1, f_2, \ldots, f_{k+1})\) and, in particular, \((\Gamma x, f_{k+1}) = 0\). Consequently,

\[
\| \Gamma \Gamma x + \Gamma f_x \|_2 = \| \beta \Gamma g_{k+1} + \Gamma f_x \|_2
\]

\[
= \| \sigma_{k+1} f_{k+1} \|_2 + \| \Gamma f_x \|_2
\]

\[
= \sigma_{k+1}^2 + \| \Gamma f_x \|_2^2
\]

However, since \(\| \Gamma \Gamma x + \Gamma f_x \|_2 \leq \| \Gamma \Gamma \|_2 \| x \|_2 = \sigma_{k+1}^2\), we conclude that

\[
\sigma_{k+1}^2 + \| \Gamma f_x \|_2^2 \leq \sigma_{k+1}^2 \Rightarrow \| \Gamma f_x \|_2 = 0 \Rightarrow \Gamma f_{k+1} = 0,
\]

so that

\[
(\Gamma \Gamma + \Gamma f_x) g_{k+1} = \sigma_{k+1} f_{k+1}.
\]

Also, if we define \(\Pi_+\) to be the stable projection operator,

\[
\sigma_{k+1} = \| \sigma_{k+1} f_{k+1} \|_2 \leq \| \Pi_+(G_0 + F) g_{k+1} \|_2
\]

\[
= \| (G_0 + F) g_{k+1} \|_2
\]

\[
= \| G_0 + F \| \| g_{k+1} \|_2
\]

\[
= \| G_0 + F \|_\infty = \sigma_{k+1},
\]

since \(F(s)\) achieves the infimum in (3.5). This shows that \(\Pi_+(G_0 + F) g_{k+1} = (G_0 + F) g_{k+1} = \sigma_{k+1} f_{k+1}\), as required.

In our application we use \(f_{k+1}(s)\) and \(g_{k+1}(s)\) as a basis for constructing two all-pass transformations to be used in a diagonalization procedure. Lemma 3.1 represents the first step in a two-stage scaling process. The aim is to find vectors \(\xi_{k+1}(s)\) and \(\psi_{k+1}(s)\) that have full rank at infinity but that retain the singular vector properties of the Schmidt pair [13, Lemmas 6, 9].
LEMMA 3.1. Let $G_0(s) = (A, B, C, D)$ be the minimal and balanced realization referred to in (2.6)-(2.8). Then

\begin{align}
 f_{k+1}(s) &\triangleq (A, e_n, C, 0), \\
 g_{k+1}(s) &\triangleq (-A^*, e_n, B^*, 0)
\end{align}

are a Schmidt pair for the Hankel operator $\Gamma_{c_0}$ corresponding to the singular value $\sigma_{k+1}$, where $e_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$. Also,

\begin{align}
 f_{k+1}(s) &= \phi_{k+1}(s) \xi_{k+1}(s), \\
 g_{k+1}(s) &= \phi_{k+1}^*(s) \psi_{k+1}(s),
\end{align}

in which

\begin{align}
 \xi_{k+1}(s) &= \begin{bmatrix} A_{11} & A_{12} \\ C_1 & C_2 \end{bmatrix}, \\
 \psi_{k+1}(s) &= \begin{bmatrix} -A_{11}^* & -A_{21}^* \\ B_1^* & B_2^* \end{bmatrix},
\end{align}

and

\begin{align}
 \sigma_{k+1} &= \det(sI - A_{11})/\det(sI - A).
\end{align}

Proof. For realizations (3.15a, b) see [3, p. 69]. Equations (3.16a, b) follow by invoking a result on partitioned determinants [7, p. 656]. See [9] for details.

To scale $\xi_{k+1}(s)$ to be of unit length, note that the scalar function $\xi_{k+1}(s)\xi_{k+1}(s)$ is positive on the imaginary axis (by 3.17a), and hence it can be spectrally factored. In particular, let $\xi_{k+1} = nd^{-1}$ be a coprime factorization over the polynomials, with $\xi_{k+1}^*\xi_{k+1} = d^{-n^*n}d^{-1}$. Factoring $n^*n = \tilde{n}^*\tilde{n}$, where $\tilde{n}$ is scalar with its zeros in $\mathbb{C}_+$, we obtain

\begin{align}
 v := \xi_{k+1}(\tilde{n}d^{-1})^{-1} = \tilde{n}^{-1} \in \mathbb{H}_{\infty, p \times p}^c,
\end{align}

with $v^*v = 1$. It is always possible to find $V_\perp(s)$ such that $V = [v \ V_\perp](s)$ is all-pass and such that $\deg(v) = \deg([v \ V_\perp])$. A similar argument may be used to derive a $w(s)$ from $\psi_{k+1}$ such that $w^*w = 1$. We summarize our progress so far.

LEMMA 3.2. Given $f \in \mathbb{H}_2$ and $g \in \mathbb{H}_2^\perp$, there exist all-pass matrices $V \in \mathbb{H}_{\infty, p \times p}$ and $W \in \mathbb{H}_{\infty, m \times m}$ given by

\begin{align}
 V &= [v \ V_\perp](s), \\
 W &= [w \ W_\perp](s),
\end{align}

in which $v$ and $w$ are given by (3.19) and its dual. Furthermore, minimal realizations of $V(s)$ and $W(s)$ are controllable from the first input.

Next, we give a concrete state-space construction of the vectors $v(s)$ and $w(s)$ along with their all-pass completions that are derived from standard spectral factorization theory.

Scaling $\xi(s) = C_2 + C_1(sI - A_{11})^{-1}A_{12}$ to unit length as $v(s) = \xi(s)m^{-1}(s)$ requires the solution of the spectral factorization problem $\xi^*(s)\xi(s) = m^*(s)m(s)$. Since $v(s)$ is required to be completely unstable (for reasons that will become apparent later), the spectral factor $m(s)$ must be nonminimum phase. This is achieved by choosing the appropriate solution to the corresponding Riccati equation [2]. The construction of the all-pass completion $V_\perp(s)$ may be achieved with no increase in degree. This accounts for the fact that minimal realizations of $V(s)$ and $W(s)$ are controllable from the first input. The construction is summarized in the following steps:
(i) If \((A_{11}, A_{12})\) is not completely controllable, perform a transformation \(T_1\) in the state space of (3.17a),

\[
\xi(s) = \begin{bmatrix}
A_{11} & A_{12} \\
C_1 & C_2
\end{bmatrix}
\overset{T_1}{\rightarrow}
\begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\
0 & \tilde{A}_{22} & 0 \\
\tilde{C}_1 & \tilde{C}_2 & 0
\end{bmatrix},
\]

in which \((\tilde{A}_{11}, \tilde{A}_{13})\) is controllable.

(ii) Choose \(\Omega_r\) as the destabilizing solution to the Riccati equation

\[
\Omega_r (\tilde{A}_{11} - \tilde{A}_{13} C_2^* \tilde{C}_1) + (\tilde{A}_{11} - \tilde{A}_{13} C_2^* \tilde{C}_1)^* \Omega_r + \Omega_r \tilde{A}_{13}^* \Omega_r - \tilde{C}_1^* C_2 \tilde{C}_1 = 0.
\]

Since \((\tilde{A}_{11}, \tilde{A}_{13})\) is controllable by construction and since the corresponding Hamiltonian is free of \(j\omega\)-axis eigenvalues (\(\xi_{n+1}(j\omega)\) is full rank \(\forall \omega \in \mathbb{R}\)), \(\Omega_r \equiv 0\) is guaranteed to exist; see [8] for more details.

(iii) \(V(s)\) is given by

\[
V(s) = \begin{bmatrix}
\tilde{A}_{11} + \tilde{A}_{13} (\Omega_r - C_2^* \tilde{C}_1) & \tilde{A}_{13} \Omega_r \tilde{C}_1 C_1^* \\
C_2 \tilde{C}_1 + C_2 \tilde{A}_{13} \Omega_r & C_2 \tilde{C}_1 + C_2 \tilde{A}_{13} \Omega_r
\end{bmatrix},
\]

Note that \([v \mid V_{\perp}]\) may also be described by the (nonminimal) realization

\[
V(s) = \begin{bmatrix}
A_{11} + A_{12} (A_{11} \Omega_r - C_2^* C_1) & A_{12} \Omega_r C_1 C_1^* \\
C_2 A_{12} \Omega_r + C_2 A_{12} \Omega_r & C_2 \Omega_r + C_2 \Omega_r C_1^*
\end{bmatrix},
\]

in which

\[
\Omega := T_1^* \begin{bmatrix}
\Omega_r & 0 \\
0 & 0
\end{bmatrix} T_1.
\]

Similarly,

\[
W(s) = \begin{bmatrix}
\tilde{A}_{31}^* - \tilde{A}_{31}^* \tilde{\Omega}_r - B_2 \tilde{B}_1^* \\
B_2 \tilde{B}_1^* + B_2 \tilde{A}_{31} \tilde{\Omega}_r
\end{bmatrix},
\]

where \(\tilde{\Omega}_r\) denotes the destabilizing solution of

\[
\tilde{\Omega}_r (\tilde{A}_{11} - \tilde{B}_1 B_1^* \tilde{A}_{31})^* + (\tilde{A}_{11} - \tilde{B}_1 B_1^* \tilde{A}_{31}) \tilde{\Omega}_r + \tilde{\Omega}_r \tilde{A}_{31} \tilde{A}_{31} \tilde{\Omega}_r - \tilde{B}_1 B_1^* + B_1 \tilde{B}_1^* = 0
\]

and the various blocks of (3.27) and (3.28) are defined by

\[
\psi(s) = \begin{bmatrix}
-A_{21}^* & -A_{21}^* \\
B_1^* & B_2
\end{bmatrix}
\overset{T_1}{\rightarrow}
\begin{bmatrix}
-A_{21}^* & -A_{21}^* \\
0 & -A_{21}^* \\
B_1^* & B_2
\end{bmatrix},
\]

in which \((\tilde{A}_{11}, \tilde{A}_{31})\) is an observable pair. It is also convenient to define

\[
\tilde{\Omega} = T_1^* \begin{bmatrix}
\tilde{\Omega}_r & 0 \\
0 & 0
\end{bmatrix} T_1
\]

in the case that \((A_{11}, A_{31})\) is not completely observable.

The construction of Lemma 3.2 together with Theorem 3.1 and Lemma 3.1 imply that \(V(s)\) and \(W(s)\) will block-diagonalize all the optimal error systems \((G_0 + F)(s)\). Moreover, the augmented system

\[
G_a(s) = \begin{bmatrix}
G_0(s) & 0 \\
0 & 0
\end{bmatrix} \in \mathbb{R}^{(p+m-1) \times (p+m-1)}
\]
will have Schmidt vectors \([f_{k+1}^T(s) \ 0]^T\) and \([g_{k+1}^T(s) \ 0]^T\) and hence will be block-diagonalized by

\[
V_\alpha(s) := \begin{bmatrix} V(s) & 0 \\ 0 & I_{m-1} \end{bmatrix}, \quad W_\alpha(s) := \begin{bmatrix} W(s) & 0 \\ 0 & I_{p-1} \end{bmatrix}.
\]

The next result establishes the required decomposition of the family of all optimal error systems.

**Lemma 3.3.** The generator of all \(k\)th-order optimal error systems \(G_\alpha + F(s)\) can be diagonalized as

\[
V^*_\alpha(G_\alpha + F)W_\alpha(s) = \begin{bmatrix} \sigma_{k+1}a(s) & 0 & 0 \\ 0 & G_1 + Q_{11} & Q_{12} \\ 0 & Q_{21} & Q_{22} \end{bmatrix},
\]

in which

(i) \(a(s)\) is all-pass (in fact, inner),

(ii) \(G_1(s) \in \mathcal{H}_+\) and \(Q(s) \in \mathcal{H}_-(k),\)

(iii) and \(G_1 + Q_{11} \in \mathcal{H}_+\) is all-pass.

**Proof.** The augmented system \(G_\alpha(s)\) has Schmidt vectors \([f_{k+1}^T(s) \ 0]^T\) and \([g_{k+1}^T(s) \ 0]^T\) corresponding to the \((k+1)th\) Hankel singular value of \(G_\alpha(s)\) and is therefore block-diagonalized by \(V_\alpha(s)\) and \(W_\alpha(s)\). The fact that \(V_\alpha(s), W_\alpha(s),\) and \((G_\alpha + F)(s)\) are all-pass implies that

\[
\frac{1}{\sigma_{k+1}(G_\alpha)} \begin{bmatrix} G_1 + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}
\]

is all-pass.

To show that \(a(s)\) is inner, one need only show that all the poles of \(F_{11}\) (which is defined in (2.14) and (2.15)) cancel when forming \(v^*(G + F_{11})w(s).\) Since we never exploit the inner character of \(a(s)\) we omit the proof.

Finally, to show that the decomposition in (ii) is valid, assume that \(V^*_\alpha(s), W_\alpha(s),\) and \(F(s)\) have minimal realizations

\[
V^*_\alpha(s) = \begin{bmatrix} v^*_\alpha \\ \vdots \\ v^*_\alpha \end{bmatrix}, \quad W_\alpha(s) = \begin{bmatrix} w \\ \vdots \\ w \end{bmatrix},
\]

and

\[
F(s) = \begin{bmatrix} A_f & B_{1f} & B_{2f} \\ C_{1f} & D_{11f} & D_{12f} \\ C_{2f} & D_{21f} & D_{22f} \end{bmatrix}.
\]
with \( V^*_u(s) \), \( W_u(s) \in \mathbb{H}^\infty \), and \( F(s) \in \mathbb{H}^\infty \). Next, let \( F_{12}(s) \) admit a right coprime factorization

\[
F_{12}(s) := nm^{-1}(s) \equiv \begin{bmatrix} A_T + B_{2f} K & B_{2f} \\ C_{1f} + D_{12f} K & D_{12f} \end{bmatrix}
\]

\[
\begin{bmatrix} A_T + B_{2f} K & B_{2f} \\ C_{1f} + D_{12f} K & D_{12f} \end{bmatrix}^{-1},
\]

in which \( K \) is chosen so that \( \lambda(A_T + B_{2f} K) \cap \lambda(A_T) = \emptyset \); Lemma A.1 in Appendix A shows that this is always possible. Next, \( v^*F_{12}(s) = 0 \Rightarrow v^*N(s) = 0 \), and we may write

\[
0 = \begin{bmatrix} A_u & B_u \\ C_{1v} & d_{1v} \end{bmatrix} \begin{bmatrix} A_T + B_{2f} K & B_{2f} \\ C_{1f} + D_{12f} K & D_{12f} \end{bmatrix}
\]

\[
\begin{bmatrix} A_u & B_u(C_{1f} + D_{12f} K) & B_{2f} \\ 0 & A_T + B_{2f} K & B_{2f} \\ C_{1v} & d_{1v}(C_{1f} + D_{12f} K) & d_{1v} \end{bmatrix}.
\]

Now let \( X \) be the (unique) solution to the linear matrix equation

\[
X(A_T + B_{2f} K) - AX + B_v(C_{1f} + D_{12f} K) = 0.
\]

Since \( V^*_u(s) \) is observable through its first output, the basis change

\[
T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}
\]

in (3.40) establishes that

\[
B_vD_{12f} + XB_{2f} = 0,
\]

which when substituted back into (3.41) gives

\[
XA_T - A_vX + B_vC_{1f} = 0.
\]

A similar argument based on \( F_{21}w(s) = 0 \) will establish the existence of a matrix \( Y \) such that

\[
YA_w - A_fY + B_{1f}C_w = 0
\]

and

\[
D_{21f}C_w - C_{2f}Y = 0.
\]

Forming the product

\[
\begin{bmatrix} V^*_u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G + F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} W_u & 0 \\ 0 & I \end{bmatrix}
\]

\[
\begin{bmatrix} A_v & B_vC & B_vC_{1f} & B_vD_{11f} & B_vD_{12f} \\ 0 & A & 0 & BC_w & BD_{2w} \\ 0 & 0 & A_f & B_{1f}C_w & B_{1f}D_{2w} \\ 0 & 0 & 0 & A_w & B_{2w} \\ C_{2u} & D_{2u}C & D_{2u}C_{1f} & D_{2u}D_{11f} & D_{2u}D_{12f} \\ 0 & 0 & C_{2f} & D_{2f} & D_{21f} \end{bmatrix}
\]

and introducing the state-space transformations

\[
T_1 = \begin{bmatrix} I & 0 & X \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & Y \\ 0 & 0 & 0 & I \end{bmatrix}
\]
together with equations (3.43)-(3.46) establishes the decomposition

$$\begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G + F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} W^* & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

in which $G_1(s) \in \mathbb{R}^\infty$ and

$$Q(s) = \begin{bmatrix} A_f & B_{1f}D_{2w} + YB_{2w} \\ D_{2e}C_{1f} - C_{2w}X \\ C_{2f} \\ 0 \end{bmatrix} \begin{bmatrix} 0 & D_{2f}D_{12f} \\ D_{2e}D_{2w} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^\infty(k).$$

This proves the result.

**Remark 3.1.** A direct calculation from the state-space formula for the scaled Schmidt vectors gives the following concrete realization for $Q(s)$ in terms of $\Omega$ and $\tilde{Q}$ defined in (3.26) and (3.30):

$$Q(s) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_1 \\ \hat{C}_1 & \sigma I \end{bmatrix} = \begin{bmatrix} A - \sigma^{-1} B_2 \hat{C}_1, \sigma I \\ -\sigma I \end{bmatrix} = \begin{bmatrix} \hat{C} & \hat{C}_1 \end{bmatrix} \begin{bmatrix} \Omega^\top & 0 \\ 0 & \sigma I \end{bmatrix} = \begin{bmatrix} \hat{C}^\top \Omega^\top + \Omega^\top \hat{C}^\top \\ \hat{C}^\top \Omega^\top + \Omega^\top \hat{C}^\top \end{bmatrix} = \begin{bmatrix} \hat{C}^\top \Omega^\top & 0 \\ \hat{C}^\top \Omega^\top + \Omega^\top \hat{C}^\top \end{bmatrix}.$$

Before we state and prove the main theorem of the section a technical result that gives certain properties of the zeros of the off-diagonal blocks of $Q(s)$ defined in (3.51) is required.

**Lemma 3.4.** Let $Q(s)$ be defined as in (3.51). Then

(i) all MacMillan zeros of $Q_{21}(s)$ and $Q_{22}(s)$ lie in the open right half plane;

(ii) if $\lambda$ is a stable eigenvalue of $A - \sigma^{-1} B_2 \hat{C}_1$, then it is an uncontrollable mode of $(\hat{A}, \hat{B}_2)$, and $-\lambda$ is a MacMillan zero of $\xi_{k+1}(s)$;

(iii) if $\lambda$ is a stable eigenvalue of $A + \sigma^{-1} \hat{B}_1 \hat{C}_2$, then it is an unobservable mode of $(\hat{A}, \hat{C}_2)$, and $-\lambda$ is a MacMillan zero of $\tilde{Q}_{k+1}(s)$.

Proof The position of the eigenvalues of $A - \sigma^{-1} B_2 \hat{C}_1$ will be established directly from (3.51), from which it is clear that they are located at the eigenvalues of

$$\Phi = \hat{A} + \Gamma^{-1} C^\top \hat{C}_1 C_1 (\Sigma + \Omega^\top \Gamma).$$

Next, we substitute (A.2) into (3.52) and make the transformation $T_1$ in (3.22) to obtain

$$T_1^* \Gamma \Phi^{-1} T_1^* = T_1^* \left[ C^\top C_1 C_1 x + C_2 A_{12} \right] T_1^* = \begin{bmatrix} \hat{C}^\top & \hat{C}^\top C_1 \hat{C}_1 \Omega^\top - \hat{A}^\top_1 + \hat{\tilde{C}}^\top_2 \hat{A}_{13} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{A}^* & \hat{A}^* C_1 \Omega^\top - \hat{A}^*_1 + \hat{\tilde{C}}^*_2 \hat{A}^*_{13} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{A}^* & \hat{A}^* C_1 \Omega^\top - \hat{A}^*_1 + \hat{\tilde{C}}^*_2 \hat{A}^*_{13} \\ 0 \end{bmatrix},$$

in which $-\hat{A}^*_2$ is completely unstable and $\Omega$ is the appropriate solution to the reduced-dimension Riccati equation (3.23). Suppose that $\exists \xi \neq 0$ and $s_0 \in \mathbb{C}_-$ such that

$$\xi^*(\hat{C}^\top C_1 \hat{C}_1 \Omega^\top - \hat{A}^*_1 + \hat{\tilde{C}}^*_2 \hat{A}^*_1) = s_0 \xi^*.$$

Then the product of $\xi^*$, the left-hand side of (3.23), and $\xi$ can be shown to imply that

$$\begin{bmatrix} A_{11} & A_{13} C_1^\top \hat{C}_1 \end{bmatrix} \xi = -s_0 \xi$$

and

$$C_1 \check{C}_1 \xi = 0.$$
Substituting from (A.2) gives
\[ (3.57) \quad \xi^* \left[ \bar{s}_0 I + \Gamma \hat{A} \Gamma^{-1} \right] C_i^C C_i^C = 0, \]
so that
\[ (3.58) \quad (\xi^* \Gamma) \left[ -\bar{s}_0 I - \hat{A} \hat{B}_2 \right] = 0. \]
This shows that \(-\bar{s}_0\) is an uncontrollable mode of \((\hat{A}, \hat{B}_2)\). Inspecting the realization of \(Q_{12}(s)\) given in (3.51), we conclude that

(i) all the MacMillan zeros of \(Q_{12}(s)\) lie in the open right half plane, and

(ii) every eigenvalue of \(\hat{A} - \sigma^{-1} \hat{B}_2 \hat{C}_i\) that lies in the left half plane is an uncontrollable mode of \([\hat{A}, \hat{B}_2]\).

Since \(s_0\) is an unstable unobservable mode of \([\hat{A}_{11} - \hat{A}_{13} C_2^C \hat{C}_1, C_1 \hat{C}_1]\), it is zero of
\[ (3.59) \quad C \]
by [12, Lemma 4.3]. Since \([\hat{A}_{11}, \hat{A}_{13}, \hat{C}_1, C_2]\) is minimal, \(s_0\) is also a MacMillan zero of \(\xi(s)\).

A dual argument will establish the position of the zeros of \(Q_{21}(s)\).

We are now in a position to present a preliminary version of the main algorithm. In this case we make a simplifying assumption about the position of the zeros of the Schmidt vectors; this restriction is removed later in Theorem 3.2.

**Theorem 3.2.** Assume that the Schmidt vectors \(\xi_{k+1}(s)\) and \(\psi^{*}_{k+1}(s)\) given in (3.17) are free of right-half-plane MacMillan zeros. Then the family \(\tilde{F}(s)\) of all approximations \(F(s, k)\) that minimize the pair
\[ (3.60) \quad (s_1^C(G_0 + F), s_2^C(G_0 + F)) \]
lexicographically is parameterized by
\[ (3.61) \quad \tilde{F}(s) = F_{11} + V_\perp (-Q_{11} + \tilde{F}_1) W^\perp(s), \]
in which \(\tilde{F}_1\) denotes the family of all optimal kth-order approximations of \(G_1(s)\), that is, every \(F_1(s)\) in \(\mathcal{H}_\infty^\infty(k)\) that satisfies \(\|G_1 + F_1\|_\infty = \sigma_{k+1}(G_1)\).

Proof. By using the results of Lemma 3.3, the family of all optimal error systems (with respect to \(s^C(\cdot)\)) may be parameterized by \(G_0 + \tilde{F}(s)\)
\[ (3.62) \quad = \left\{ V(s)[(\sigma_{k+1}(G_0) a(s)) \Theta(G_1 + \tilde{F}_1(Q, \Theta))] W^*(s) \right\} \Theta(s) \in \frac{1}{\sigma_{k+1}(G_0)} \mathcal{B} \mathcal{H}_\infty^\infty \}
Since \(V(s), W^*(s),\) and \(a(s)\) are all-pass, the family of all approximations that minimize \(s_2(G_0 + \tilde{F})\) are generated by
\[ (3.63) \quad \inf_{\Theta \in 1/\sigma_{k+1}(G_0) \mathcal{B} \mathcal{H}_\infty^\infty} \|G_1 + \tilde{F}_1(Q, \Theta)\|_\infty = s_2^C(G_0 + \tilde{F}). \]
It will now be shown that if \(\xi_{k+1}(s)\) and \(\psi^{*}_{k+1}(s)\) have no MacMillan zeros in the right half plane, the minimization problem in (3.63) is equivalent to the unconstrained problem
\[ (3.64) \quad \inf_{\tilde{F}_1 \in \mathcal{H}_\infty^\infty(l)} \|G_1 + \tilde{F}\|_\infty = \sigma_{l+1}(G_1), \]
where \(l\) is the MacMillan degree of the stable part of \(Q(s)\). Since \(Q(s) \in \mathcal{H}_\infty^\infty(l)\) and \(\|Q_{22}(\Theta)\|_\infty < 1\), a small-gain argument [6], [11] shows that \(\tilde{F}_1(Q, \Theta) \in \mathcal{H}_\infty^\infty(l)\) for all \(\Theta(s) \in 1/\sigma_{k+1}(G_0) \mathcal{B} \mathcal{H}_\infty^\infty\). By comparing (3.63) and (3.64) we may write
\[ (3.65) \quad \inf_{\Theta(s) \in 1/\sigma_{k+1}(G_0) \mathcal{B} \mathcal{H}_\infty^\infty} \|G_1 + \tilde{F}_1(Q, \Theta)\|_\infty \geq \inf_{\tilde{F}_1 \in \mathcal{H}_\infty^\infty(l)} \|G_1 + \tilde{F}\|_\infty = \sigma_{l+1}(G_1). \]
It will be shown that every $\tilde{F} \in \mathcal{H}_c^\infty (I)$ that achieves the minimum in (3.64) is generated by some $\Theta (s) \in 1/\sigma_{k+1} (G_0) \mathcal{B} \mathcal{H}_c^\infty$ through $\mathcal{F}_I (Q, \Theta)$. This is based on an argument given in [5].

Since all the MacMillan zeros of $Q_{12} (s)$ lie in the open right half plane (Lemma 3.4), $Q_{12} (j\omega)$ is nonsingular for every $\omega \in \mathbb{R}$. This and the all-pass character of

\begin{equation}
\frac{1}{\sigma_{k+1} (G_0)} \begin{bmatrix}
G_1 + Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix}
\end{equation}

give

\begin{equation}
\| G_1 + Q_{11} \|_\infty < \sigma_{k+1} (G_0).
\end{equation}

Furthermore,

\begin{equation}
\sigma_{l+1} (G_1) \equiv \| G_1 + Q_{11} \|_\infty,
\end{equation}

so that

\begin{equation}
\sigma_{l+1} (G_1) < \sigma_{k+1} (G_0).
\end{equation}

In addition, we will demonstrate that

\begin{equation}
\{ \mathcal{F}_I (Q, \Theta) : \| \Theta (s) \|_\infty < 1/\sigma_{k+1}, \Theta \in \mathcal{H}_c^\infty \}
\end{equation}

generates every $\tilde{F} (s) \in \mathcal{H}_c^\infty (I)$ with the property $\| G_1 + \tilde{F} \|_\infty < \sigma_{k+1} (G_0)$. Suppose that some $\tilde{F} (s)$ satisfies $\| G_1 + \tilde{F} \|_\infty < \sigma_{k+1} (G_0)$. Then

\begin{equation}
\tilde{\Theta} (s) = \mathcal{F}_I \left( \frac{1}{\sigma_{k+1}^2 (G_0)} \begin{bmatrix}
Q_{22}^* & Q_{12}^* \\
Q_{21}^* & G_1^* + Q_{11}^*
\end{bmatrix}, G_1 + \tilde{F} (s) \right)
\end{equation}

gives

\begin{equation}
\tilde{F} (s) = \mathcal{F}_I \left( \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix}, \tilde{\Theta} (s) \right).
\end{equation}

Since $\| G_1 + Q_{11} \|_\infty < \sigma_{k+1} (G_0)$ and $\| G_1 + \tilde{F} \|_\infty < \sigma_{k+1} (G_0)$, the linear fractional map defining $\tilde{\Theta} (s)$ is well posed and, furthermore, $\| \tilde{\Theta} (s) \|_\infty < \sigma_{k+1}^{-1} (G_0)$. It remains to be shown that $\tilde{\Theta} (s) \in \mathcal{H}_c^\infty$. Suppose, contrary to what will be proved, that $\tilde{\Theta} (s)$ has $r$ poles in the left plane. This implies that the $A$-matrix for $\mathcal{F}_I (Q, \tilde{\Theta})$ has $l + r$ eigenvalues in the left half plane by a Nyquist-type argument [6], [14]. Since $\tilde{F} \in \mathcal{H}_c^\infty (I)$ by assumption, there must be at least $r$ cancellations in the closed loop. Since any cancellation is constrained to occur at a zero of (3.52a) or (3.52b) (by [10, Thm. 4.3]), we obtain the required contradiction, since the zero of (3.52a) and (3.52b) are all in the open right half plane by Lemma 3.4 and the theorem's hypothesis.

It follows, therefore, that all optimal $l$th order approximations of $G_1 (s)$ are generated through $\mathcal{F}_I (Q, \Theta)$, as claimed. As a consequence, the set of all $F (s)$'s that minimize $(s_1^* (G_0 + F), s_2^* (G_0 + F))$ lexicographically is parameterized by

\begin{equation}
\tilde{\mathcal{F}} (s) = -G_0 + V \begin{bmatrix}
\sigma_{k+1} (G_0) a (s) & 0 \\
0 & G_1 (s) + \mathcal{F}_I (s)
\end{bmatrix} W^*
\end{equation}

\begin{equation}
= -G_0 + V \begin{bmatrix}
\sigma_{k+1} (G_0) a (s) & 0 \\
0 & G_1 + Q_{11}
\end{bmatrix} W^* + V \begin{bmatrix}
0 & 0 \\
0 & \mathcal{F}_I - Q_{11}
\end{bmatrix} W^*
\end{equation}

where $\tilde{\mathcal{F}}_I (s)$ denotes the family of all $l$th-order optimal approximations of $G_1 (s)$. Finally, by Lemma 3.4 and the theorem's hypothesis, all stable modes of $A$ are
controllable through $\hat{B}_2$ and observable through $\hat{C}_2$, and so $l = \deg (Q) = \deg ([F_{22}]) = k$. This completes the proof of the theorem.

**Remark 3.2.** Theorem 3.2 establishes that all solutions that minimize $s_2^\infty (G_0 + F)$ can be parameterized in terms of all solutions that minimize $s_1^\infty (G_1 + F_1)$, which is a problem of dimension $(p-1) \times (m-1)$. The procedure can now be continued until a single-input or single-output problem is encountered (or until $\deg ([F_{i+1}]) \leq \deg ([Q_{i+1}])$]. At this point there is a unique optimal approximation and the process stops.

If the assumptions on the MacMillan zeros of the Schmidt vectors $\xi_{k+1}(s)$ and $\psi_{k+1}(s)$ in Theorem 3.2 are relaxed, the construction of Theorem 3.2' will generate the family of superoptimal approximations with respect to the first two singular values.

**Theorem 3.2'.** (i) Let $Q(s)$ be defined as in (3.51). Then its off-diagonal blocks $Q_{12}(s)$ and $Q_{21}(s)$ may be factored as $Q_{12}(s) = B(s)Q_{12}(s)$ and $Q_{21}(s) = Q_{21}(s)A(s)$, where $B(s)$ and $A(s)$ are inner and have degrees equal to the number of modes in $\hat{O}(\hat{A}, \hat{B}_2) \cap C(\hat{A}, \hat{B}_2) \cap C_- \cap O(\hat{A}, \hat{C}_2) \cap \hat{C}(\hat{A}, \hat{B}_2) \cap C_- \cap C_-$, respectively.

(ii) The family $\mathcal{F}(s)$ of all approximations $F \in \mathcal{H}_\infty^-(k)$ that minimize the pair

$$s_1^\infty (G_0 + F), s_2^\infty (G_0 + F)$$

lexicographically is parameterized by

$$\mathcal{F}(s) = F_1 + V_1 \mathcal{B}(\mathcal{F}_1 - [\bar{Q}_{11}]_+ - [A^*(G_1 + Q_{11})A^*]_+ \mathcal{A}W_1(s),$$

in which $\mathcal{F}_1$ denotes the family of all optimal $l$th-order Hankel-norm approximations to $[A^*(G_1 + Q_{11})A^* - \bar{Q}_{11}]_+$, where $l = \deg ([F_{22}]) \leq k$ and $\bar{Q}_{11} \in \mathcal{H}_\infty^-(I)$ is defined in the proof.

**Proof.** We begin by putting $(\hat{A}, \hat{B}_2, \hat{C}_2)$ (defined in (3.51)) into the Kalman canonical form

$$Q(s) = \begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} & 0 & \hat{B}_{12} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} & \hat{B}_{22} \\ 0 & 0 & \hat{A}_{33} & 0 & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44} & 0 \\ \hat{C}_{21} & 0 & \hat{C}_{23} & 0 & 0 \end{bmatrix},$$

so that

$$\begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} & 0 & \hat{B}_{11} & \hat{B}_{12} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} & \hat{B}_{21} & \hat{B}_{22} \\ 0 & 0 & \hat{A}_{33} & 0 & \hat{B}_{31} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44} & \hat{B}_{41} & 0 \\ \hat{C}_{11} & \hat{C}_{12} & \hat{C}_{13} & \hat{C}_{14} & 0 & -\sigma I \\ \hat{C}_{21} & 0 & \hat{C}_{23} & 0 & \sigma I & 0 \end{bmatrix}.$$

Next, we transform the controllable realization $(\hat{A}_{22}, [\hat{B}_{21}, \hat{B}_{22}], \hat{C}_{12})$ into

$$\begin{bmatrix} \hat{A}_{12} & \hat{A}_{12} & \hat{A}_{12} & \hat{B}_{12} & \hat{B}_{22} \\ 0 & \hat{A}_{22} & 0 & \hat{B}_{21} & \hat{B}_{22} \\ 0 & \hat{A}_{22} & \hat{B}_{22} & \hat{B}_{22} & \hat{B}_{22} \\ 0 & \hat{A}_{22} & \hat{B}_{22} & \hat{B}_{22} & \hat{B}_{22} \\ \hat{C}_{12} & \hat{C}_{12} & \hat{C}_{12} & \hat{C}_{12} & 0 \end{bmatrix}.$$
where the three partitions correspond to the unobservable modes ($\hat{\mathbf{A}}_{12}^{11}$) and the stable and unstable unobservable modes ($\hat{\mathbf{A}}_{12}^{22}$ and $\hat{\mathbf{A}}_{22}^{22}$, respectively). We also note that $\hat{\mathbf{A}}_{22}^{22}$ is completely unstable by Lemma 3.4. A similar transformation is carried out on $(\hat{\mathbf{A}}_{33}, \hat{\mathbf{B}}_{31}, [\hat{\mathbf{C}}_{13}^T \hat{\mathbf{C}}_{23}^T]^T)$. Combining these gives

$$Q(s) = \begin{bmatrix} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{13}^T & \hat{\mathbf{B}}_{11} & \hat{\mathbf{B}}_{12} \\ \hat{\mathbf{A}}_{21} & \hat{\mathbf{A}}_{23}^T & \hat{\mathbf{B}}_{21} & \hat{\mathbf{B}}_{22} \\ \hat{\mathbf{A}}_{31} & \hat{\mathbf{A}}_{33}^T & \hat{\mathbf{B}}_{31} & \hat{\mathbf{B}}_{32} \\ \hat{\mathbf{C}}_{11} & \hat{\mathbf{C}}_{13}^T & \hat{\mathbf{C}}_{14} & 0 & -\sigma I \\ \hat{\mathbf{C}}_{21} & \hat{\mathbf{C}}_{23}^T & \hat{\mathbf{C}}_{24} & 0 & \sigma I & 0 \end{bmatrix}$$

Next, we consider the minimal realization

$$[Q_{21}|Q_{22}] = \begin{bmatrix} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{13} & \hat{\mathbf{B}}_{11} & \hat{\mathbf{B}}_{12} \\ 0 & \hat{\mathbf{A}}_{33} & \hat{\mathbf{B}}_{31} & 0 \\ \hat{\mathbf{C}}_{21} & \hat{\mathbf{C}}_{23}^T & \sigma I & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{A}}_{13} & \hat{\mathbf{B}}_{13} & 0 & \hat{\mathbf{B}}_{12} \\ 0 & \hat{\mathbf{B}}_{31} & 0 & \hat{\mathbf{B}}_{32} \\ \hat{\mathbf{C}}_{11} & \hat{\mathbf{C}}_{13}^T & \hat{\mathbf{C}}_{14} & 0 & -\sigma I \\ \hat{\mathbf{C}}_{21} & \hat{\mathbf{C}}_{23}^T & \hat{\mathbf{C}}_{24} & 0 & \sigma I & 0 \end{bmatrix},$$

in which we may assume without loss of generality (by possibly redefining $\hat{\mathbf{A}}_{13}$, $\hat{\mathbf{A}}_{33}$, $\hat{\mathbf{C}}_{23}$, $\hat{\mathbf{B}}_{11}$, and $\hat{\mathbf{B}}_{12}$) that $\mathcal{A}(s)$ is inner; details of this factorization appear in Appendix B. Similarly, a (left) inner factor $\mathcal{B}(s)$ may be extracted from $Q_{12} = \mathcal{B}(s)\overline{Q}_{12}$, leading to a realization

$$[\overline{Q}_{12}|\overline{Q}_{22}] = \begin{bmatrix} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{13} & \hat{\mathbf{B}}_{11} & \hat{\mathbf{B}}_{12} \\ 0 & \hat{\mathbf{A}}_{33} & \hat{\mathbf{B}}_{31} & 0 \\ \hat{\mathbf{C}}_{21} & \hat{\mathbf{C}}_{23}^T & \hat{\mathbf{C}}_{24} & 0 & -\sigma I \end{bmatrix} \begin{bmatrix} \hat{\mathbf{A}}_{13} & \hat{\mathbf{B}}_{13} & 0 & \hat{\mathbf{B}}_{12} \\ 0 & \hat{\mathbf{B}}_{31} & 0 & \hat{\mathbf{B}}_{32} \\ \hat{\mathbf{C}}_{11} & \hat{\mathbf{C}}_{13}^T & \hat{\mathbf{C}}_{14} & 0 & \sigma I & 0 \end{bmatrix},$$

in which $\hat{\mathbf{A}}_{21}$, $\hat{\mathbf{B}}_{22}$, and $\hat{\mathbf{C}}_{11}$ may have been redefined. Next, we define

$$\tilde{Q}(s) = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{13} & \hat{\mathbf{B}}_{11} & \hat{\mathbf{B}}_{12} \\ 0 & \hat{\mathbf{A}}_{33} & \hat{\mathbf{B}}_{31} & 0 \\ \hat{\mathbf{C}}_{11} & \hat{\mathbf{C}}_{13}^T & \hat{\mathbf{C}}_{14} & 0 & -\sigma I \\ \hat{\mathbf{C}}_{21} & \hat{\mathbf{C}}_{23}^T & \hat{\mathbf{C}}_{24} & 0 & \sigma I & 0 \end{bmatrix},$$

Now, \{OHLP system zeros of the realization of $Q_{21}$ in (3.79)\} $\subseteq \lambda(\hat{\mathbf{A}}_{22}^2) \cup \lambda(\hat{\mathbf{A}}_{44})$. 
where OLHP denotes the open left half plane, implies that

\[ \{ \text{System zeros of the realization of } Q_{21} \text{ in (3.80)} \} \subseteq \mathbb{C}_+. \]

As a result of the all-pass factorization (3.81),

\( \{ \text{System zeros of the realization of } \tilde{Q}_{21} \text{ in (3.83)} \} \subseteq \{ \text{System zeros of the realization of } Q_{12} \text{ in (3.80)} \} \cup \lambda(\hat{A}^u_{22}). \)

We conclude that

\( \{ \text{System zeros of the realization of } \tilde{Q}_{21} \text{ in (3.83)} \} \subseteq \mathbb{C}_+. \)

A dual argument shows that the system zeros of the realization of \( Q_{12} \) in (3.83) also lie in the open right half plane. Thus for any \( \sigma \Theta \in \mathcal{B}\mathcal{H}_\infty^c \)

\[
s_2^\infty(G_0 + \mathcal{F}) = \| G_1 + \mathcal{F}_l(Q, \Theta) \|_{\infty} \\
= \| G_1 + Q_{11} + Q_{12}\Theta(I - Q_{22}\Theta)^{-1}Q_{21} \|_{\infty} \\
= \| \mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^* - \tilde{Q}_{11} + \mathcal{F}_l(\tilde{Q}, \Theta) \|_{\infty} \\
(3.84) = \| \mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^* - \tilde{Q}_{11} \|_{\infty} + \mathcal{F}_l(\tilde{Q}, \Theta) \|_{\infty} \\
\geq \sigma_{l+1}[\mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^* - \tilde{Q}_{11}],
\]

in which \( J(s) := \mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^* - \tilde{Q}_{11} \). We may now use the arguments of Theorem 3.2 to show that

\[
\mathcal{F}_l(\begin{bmatrix} J & 0 \\
0 & 0 \end{bmatrix} + \tilde{Q}, \Theta)
\]
generates all \( \sigma_{l+1} \)-suboptimal approximations of \( [\mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^* - \tilde{Q}_{11}]_+ \) in \( \mathcal{H}_\infty^c(l) \).

To do this we note the following:

(i) The matrix

\[
(3.85) \frac{1}{\sigma} \left\{ \mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^* - \tilde{Q}_{11} \right\} + \tilde{Q}
\]
is all-pass.

(ii) \( \| \tilde{Q}_{22} \|_\infty = \| F_{22} \|_\infty < \sigma \) and the system zeros of the off-diagonal blocks of

\[
(3.86) \begin{bmatrix} J & 0 \\
0 & 0 \end{bmatrix} + \tilde{Q}
\]
lie in the open right halfplane. This follows from the fact that \( J(s) \) is completely unstable.

(iii)

\[
(3.87) \tilde{Q} \in \mathcal{H}_\infty^c(l), \quad \text{where } l = \text{deg } ([F_{22}]_+).
\]

It follows that

\[
(3.88) \inf_{\omega \in \mathbb{R}} \{ s_2^\infty(G_0 + \mathcal{F}) \} = \sigma_{l+1}([\mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^* - \tilde{Q}_{11}]_+).
\]
Finally, the family of all approximations in $\mathcal{H}_\infty^-(k)$ that minimize the first two singular values is obtained by back substitution as

$$\mathcal{F}(s) = F_{11} + V_+ B(F_1 - [\hat{Q}_{1}])_+ - [\mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^*]_+ A^+ (s),$$

where $\mathcal{F}$ denotes the family of all optimal $l$th-order Hankel-norm approximations to $[\mathcal{B}^*(G_1 + Q_{11})\mathcal{A}^* - \hat{Q}_{1}].$

A simple inductive generalization of Theorem 3.2' will establish the following algorithm for calculating the (unique) superoptimal approximation.

**Algorithm 3.1.**

Given any $G_0(s) \in \mathcal{H}_\infty^+p \times m$, this algorithm finds $F_{so}(s) \in \mathcal{H}_\infty^-p \times m(k)$, which is the superoptimal $k$th-order approximation of $G_0(s)$.

1. $r = \text{rank}(G_0) \leq \min(p, m)$
2. Find $F(s)$ for $G_0(s)$ (the generator of all optimal $k$-th-order approximations—use state-space formulas in (2.15)); set $F_0 = F$
3. $F_{so} = F_{11}$
4. $V = I_m$ and $W = I_p$
5. For $i = 0$ to $r - 1$
   1. Find $\xi_i(s)$ and $\psi_i(s)$ corresponding to $G_i(s)$, and construct $v_i(s)$, $w_i(s)$, $V_+(s)$, and $W_+(s)$ using state-space formulas (3.24) and (3.27)
   2. Define $G_{i+1}(s) \in \mathcal{H}_\infty^+$ and $Q_{i+1}(s) \in \mathcal{H}_\infty^-(k)$, through decompositions given in Lemma 3.3
   3. IF $\xi_i(s)$ and/or $\psi_i(s)$ have right-half-plane MacMillan zeros THEN
      1. Extract all-pass common factors $\mathcal{A}(s)$ and $\mathcal{B}(s)$ (see Theorem 3.2')
      2. Redefine $V_+(s) := V_{i+1}(s) \mathcal{B}(s)$, $W_+(s) := \mathcal{A}(s) W_+(s)$, $Q_{i+1}(s) := [\hat{Q}_{i+1}]_+ + [\mathcal{B}^*(G_{i+1} + Q_{i+1}^o)\mathcal{A}^*]_+ - [\mathcal{B}^*(G_{i+1} + Q_{i+1}^o)\mathcal{A}^* - \hat{Q}_{i+1}]_+$, $G_{i+1}(s) := [\mathcal{B}^*(G_{i+1} + Q_{i+1}^o)\mathcal{A}^*- Q_{i+1}]_+$
   ELSE $\hat{Q}_{i+1}(s) := Q_{i+1}(s)$
4. $l = \deg[[F_{22}^o]]$
5. Find $F_{i+1}$, the generator of all $l$th-order optimal approximations to $G_{i+1}$ (using (2.15))
6. $V = V_{i+1}$ and $W = W_{i+1}$
7. $F_{so}(s) = F_{so} + V(F_{i+1}^o - \hat{Q}_{i+1}) W^o(s) \in \mathcal{H}_\infty^-(k)$

Note that if the above algorithm is terminated after $l < r$ steps, the first $l+1$ singular values will be minimized and

$$F_{so}^{(l)}(s) = F_{11}(s) + \sum_{j=0}^{l} \left\{ \prod_{i=0}^{j} V_+(s) (-Q_{i+1} + F_{i+1}) \left( \prod_{i=0}^{j} W_+(s) \right)^o \right\}(s)$$

will have no more than $k$ poles in the open left half plane.

**Remark 3.3.** In the usual case for which the realizations of $f_{k+1}$ and $g_{k+1}$ in (3.15) are free from left-half-plane zeros, a pole-zero cancellation analysis similar to the one carried out in [9] establishes the existence of extensions that are optimal with respect to the first two singular values and are such that $\deg(F_{so}^{(2)}) \leq 2n - 3$. If this assumption holds for all $G_i$'s generated by Algorithm 3.1, it can be shown that

$$\deg(F_{so}) \leq \sum_{i=1}^{\text{rank}(G_0)} (n - i).$$

The cancellation analysis is moderately intricate and is consequently omitted.
Example 3.1. In this example we illustrate a number of the features of Algorithm 3.1. Suppose \( G(s) = C(sI - A)^{-1}B \) is given by

\[
A = \begin{bmatrix}
-2\mu(\mu^2 - 1)^{-1} & 2(\mu^2 - 1)^{-1} & a_{13} \\
2(\mu^2 - 1)^{-1} & -1 & -1 \\
a_{31} & 2 & -2
\end{bmatrix},
\]

\[
B = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & b_{12} \\
\sqrt{2} & 0 \\
-3\sqrt{2}/4 & \frac{1}{\sqrt{7/2}}
\end{bmatrix},
\]

\[
C^T = \frac{1}{\sqrt{2}} \begin{bmatrix}
\sqrt{2}\mu/(\mu^2 - 1) & -\sqrt{2}\mu/(\mu^2 - 1) \\
1 & 1 \\
1 & -1
\end{bmatrix},
\]

in which \( \mu > 1 \) and

\[
a_{13} = \frac{19\mu^2 - 3 - \sqrt{7}(4\mu^2 - (\mu^2 - 1)^2)}{\sqrt{2}(\mu^2 - 1)(1 - 4\mu^2)},
\]

\[
a_{31} = \frac{\sqrt{2}\mu(-1 - 3\mu^2 + \sqrt{7}(4\mu^2 - (\mu^2 - 1)^2))}{(\mu^2 - 1)(1 - 4\mu^2)},
\]

\[
b_{12} = \frac{\sqrt{4\mu^2 - (\mu^2 - 1)^2}}{\mu^2 - 1}.
\]

It may be verified that (3.91) is balanced with controllability and observability gramians

\[
P = Q = \Sigma = \text{diag} (\mu/2, 0.5, 0.25).
\]

By applying the triangularizing transformation

\[
T = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2\mu(\mu^2 - 1)^{-1} & 0 & 1
\end{bmatrix}
\]

we conclude that the scaled Schmidt vector \( \xi_3(s) = C_2 + C_1(sI - A_{11})^{-1}A_{12} \) corresponding to \( \sigma_3 = 0.25 \) has a single MacMillan zero in the open right half plane for every \( \mu > 1 \). In the remainder of this example we fix \( \mu \) at \( \mu = 1.2 \) and seek the superoptimal Hankel-norm approximation to \( G(s) \) over \( \mathcal{H}_\infty^c(2) \).

A realization of \( F(s) \) (defined in (2.15)) that generates all optimal approximations with respect to the first singular value may be calculated as

\[
F(s) \triangleq \begin{bmatrix}
-11.196 & 0 & 3.8569 & 5.5031 & 0 \\
-5.0997 & -1.6667 & 2.6667 & 0 & -1.3333 \\
-1.2868 & -0.4861 & 0.1326 & -0.1169 & -0.1768 \\
1.2868 & -0.2210 & -0.1326 & 0.1169 & -0.1768 \\
-0.8278 & -0.1654 & 0.1654 & 0.1875 & 0
\end{bmatrix},
\]

in which we note that a stable mode (\( \lambda = -11.196 \)) is uncontrollable through the last column of the \( B \)-matrix (\( \hat{B}_2 \)). The decomposition of Lemma 3.3 is now carried out to
give

\[ Q(s) = \begin{bmatrix}
-11.196 & 0 & 8.0841 & 0 \\
-5.0997 & -1.6667 & 1.9608 & -1.3333 \\
0 & -0.5767 & 0 & -0.2500 \\
-0.8278 & -0.1654 & 0.2500 & 0
\end{bmatrix} \]

and

\[ G_1(s) = \begin{bmatrix}
-0.9485 & -2.6797 & -2.6388 \\
1.7808 & -11.662 & 4.7682 \\
0.7585 & 0.13606 & 0
\end{bmatrix} \]

which may be written in transfer function form as

\[ Q(s) = \begin{bmatrix}
\frac{1.3727(s + 32.221)}{(s + 11.196)(s + 1.6667)} & \frac{0.25(s - 1.4142)}{s + 1.6667} \\
0.25(s - 1.4142)(s - 11.196) & 0.22048
\end{bmatrix} \]

and

\[ G_1(s) = \frac{1.3532(s + 24.434)}{(s + 11.196)(s + 1.4142)}. \]

Note that because of the presence of a minimum-phase system zero in the realization of \( Q_{12}(s) \) in (3.97) (due to the uncontrollable mode \( \lambda = -11.198 \)), \( \mathcal{F}(Q, \sigma_3^{-1} \mathcal{B} \mathcal{H}^\infty) \) does not generate all the \( \sigma_3 = 0.25 \)-suboptimal approximations of \( G(s) \) in \( \mathcal{H}^\infty \). We may now extract the inner factor

\[ \mathcal{A}(s) = \frac{s - 11.196}{s + 11.196} \]

corresponding to this zero, so that

\[ G_1(s) + \mathcal{F}(Q, \sigma_3^{-1} \Theta) = (\tilde{G}_1 + \mathcal{F}(\tilde{Q}, \sigma_3^{-1} \Theta)) \mathcal{A}(s), \]

in which

\[ \tilde{G}_1 := -\frac{1.3532(s + 24.434)}{(s - 11.196)(s + 1.4142)} \]

and

\[ \tilde{Q}(s) = \begin{bmatrix}
\frac{1.3727(s + 32.221)}{(s - 11.196)(s + 1.6667)} & \frac{0.25(s - 1.4142)}{s + 1.6667} \\
0.25(s - 1.4142) & 0.22048
\end{bmatrix}. \]

Furthermore, it may be verified that

\[ \mathcal{F}(\tilde{Q} + \begin{bmatrix} \tilde{G}_1 & 0 \\ 0 & 0 \end{bmatrix}, \sigma_3^{-1} \mathcal{B} \mathcal{H}^\infty) \]

generates all \( \sigma_3 \)-suboptimal approximations of \( [\tilde{G}_1]_+ \) in \( \mathcal{H}^\infty(1) \), so that

\[ s_2^\infty(G + F) = \inf_{F_1 \in \mathcal{H}^\infty(1)} \|[\tilde{G}_1]_+ + F_1 \|_\infty = 0, \]
which is in agreement with Remark 2.2. The superoptimal approximation is finally obtained as

$$F_{so} = F_{11} - V_\perp(G_1 + Q_{11})W^*_\perp$$

(3.107)

$$= \begin{bmatrix}
-11.244 & 0.5310 & 0 & 4.0836 & 5.6906 \\
-8.8849 & -1.3664 & 0 & 0.9722 & -1.2665 \\
0 & 0 & 1.4142 & -0.1467 & -0.0541 \\
-1.3554 & -0.6586 & -0.1297 & 0.1326 & -0.1169 \\
1.1951 & -0.3677 & 0.3130 & -0.1326 & 0.1169 \\
\end{bmatrix}.$$  

4. Conclusions. The purpose of this paper is to develop an implementable state-space version of Young's algorithm for superoptimal Hankel-norm approximations [16]. The new Algorithm 3.1 requires only standard linear algebraic library routines and has the added advantage that it can be stopped after only $l < r$ steps. In this event, the procedure produces a representation formula for all the approximations in $H^\infty(k)$ that minimize $\{s(G_0 + \mathcal{P}), s^*(G_0 + \mathcal{P}), \ldots, s^*(G_0 + \mathcal{P})\}$ with respect to lexicographic ordering.

Appendix A.

Lemma A.1. Let $\tilde{A}$ denote the $A$-matrix of a minimal realization of $V^*(s)$. Then there exists a matrix $K$ such that $\lambda(A_f + B_{zf}K) \cap \lambda(\tilde{A}) = \emptyset$.

Proof. Since $\tilde{A}$ is stable, it suffices to show that every stable eigenvalue of $A_f$ that is uncontrollable through $B_{zf}$ does not belong to the spectrum of $\tilde{A}$. Suppose for contradiction that $\lambda$ is such an eigenvalue. Then $\exists \beta \neq 0$ such that

$$\beta^*[\lambda I - A_f | B_{zf}] = 0.$$  

It follows by using (2.15), (2.6), and (2.7) that

$$\Gamma^{-1}A^* \Gamma + \Sigma \Gamma^{-1}C^* C \Gamma C = \Gamma^{-1}(\sigma^2 I + \Sigma A_{11} + \Sigma - \sigma B_1 U^* C_1)\Gamma$$

$$+ \Sigma \Gamma^{-1}C^* (I - C_2 C^* \Sigma - \Sigma B_1 U^* C_1)\Gamma$$

(4.2)

$$= \Gamma^{-1}(\sigma^2 A_{11} + \sigma B_1 B^* \Sigma^2 C_1 - \Sigma^2 A_{11} - \Sigma C_1 C - C_{12} C_{13})$$

$$= -A_{11} + A_{12} C_{13} C_1.$$  

Substituting (4.2) into (4.1) yields

$$\tilde{\beta}^*[\lambda I + A_{11} + C_{12} A_{12}^* C_{13} + \Sigma^* C_{13}^*] = 0,$$

where $\tilde{\beta} = \Gamma^{-1} \beta$. Next, we introduce the transformation $T_1$, defined in (3.22), to (4.3). This gives

$$[\hat{\beta}^* \ \hat{\gamma}^*] \begin{bmatrix}
\lambda I + \hat{A}_{11}^* - \hat{C}_{12}^* C_2 \hat{A}_{13} & 0 & \hat{C}_{13}^* C_{13}^* \\
\hat{A}_{12}^* - \hat{C}_{12}^* C_2 \hat{A}_{13} & \lambda I + \hat{A}_{12}^* & \hat{C}_{13}^* C_{13}^* \\
\end{bmatrix} = 0,$$

which implies that

$$\hat{\beta}^*(\lambda I + \hat{A}_{11}^* - \hat{C}_{12}^* C_2 \hat{A}_{13}) + \hat{\gamma}^*(\hat{A}_{12}^* - \hat{C}_{12}^* C_2 \hat{A}_{13}) = 0,$$

$$\hat{\gamma}^*(\lambda I + \hat{A}_{12}^*) = 0,$$

$$\hat{\gamma}^* \hat{C}_{13} + \hat{\beta}^* \hat{C}_{13}^* = 0.$$  

Now, since $\lambda \in \mathbb{C}_-$ and $\lambda(-\hat{A}_{12}^*) \subseteq \lambda(-A_{11}^*) \subseteq \mathbb{C}_+$,

$$\hat{\beta}^* = 0 \quad \Rightarrow \quad \hat{\beta}^* \neq 0.$$
Thus
\[(A.9) \quad (A.5) \Rightarrow \hat{\beta}_1^*(\lambda I + \hat{A}_{11}^* - \hat{C}_1^* \hat{C}_1) = 0,\]
\[(A.10) \quad (A.7) \Rightarrow \hat{\beta}_1^* \hat{C}_1^* = 0.\]
From the multiplication of \(\hat{\beta}_1^*,\) the left-hand side of (3.23), and \(\hat{\beta}_1\) we obtain
\[(A.11) \quad -2 \text{Re}(\lambda) \hat{\beta}_1^* \hat{\Omega}_1 \hat{\beta}_1 + \hat{\beta}_1^* \hat{\Omega}_1 \hat{A}_{11}^* \hat{A}_{11}^* \hat{\beta}_1 = 0 \Rightarrow \hat{\Omega}_1 \hat{\beta}_1 = 0,\]
since \(\text{Re}(\lambda) < 0.\) It follows from (3.24) that
\[(A.12) \quad \nu^*(s) = \left[ -\hat{A}_{11}^* - (\Omega_1 \hat{A}_{11}^* - \hat{C}_1^* \hat{C}_2) \hat{A}_{11}^* \quad \hat{C}_1^* \hat{C}_1^* \hat{C}_1^* + \hat{\Omega}_1 \hat{A}_{11}^* \hat{C}_2 \right],\]
and it is easy to show by using (A.11) that \(\lambda\) is an uncontrollable mode of this realization, which proves the result.

Appendix B.

**Lemma B.1.** \([Q_{21} | Q_{22}]\) in (3.80) may be factored as
\[(B.1) \quad \left[ \begin{array}{ccc}
\hat{A}_{11} & \hat{A}_{13} & \hat{B}_{11} \\
0 & \hat{A}_{13} & \hat{B}_{12} \\
\hat{C}_{21} & \hat{C}_{23} & 0 \\
\end{array} \right] \ast \left[ \begin{array}{ccc}
\hat{A}_{33} & \hat{B}_{31} & 0 \\
\sigma^{-1} \hat{C}_{23} & I & 0 \\
0 & 0 & I \\
\end{array} \right],\]
in which \(\mathcal{A}(s) := (\hat{A}_{11}, \hat{B}_{11}, \sigma^{-1} \hat{C}_{23}, I)\) is inner.

**Proof.** Consider the minimal realization
\[(B.2) \quad \left[ \begin{array}{ccc}
\hat{A}_{11} & \hat{A}_{13} & \hat{B}_{11} \\
0 & \hat{A}_{13} & \hat{B}_{12} \\
\hat{C}_{21} & \hat{C}_{23} & 0 \\
\end{array} \right] \ast \left[ \begin{array}{ccc}
\hat{A}_{33} & \hat{B}_{31} & 0 \\
\sigma^{-1} \hat{C}_{23} & I & 0 \\
0 & 0 & I \\
\end{array} \right],\]
with controllability grammian
\[(B.3) \quad P = \left[ \begin{array}{cc}
P_{11} & P_{12} \\
P_{12} & P_{22} \end{array} \right],\]
in which \(\dim(P_{11}) = \dim(\hat{A}_{11}) + \dim(\hat{A}_{13}).\) We will assume the following without loss of generality:

(i) \(P_{12} = 0.\) This may be achieved by introducing the state-space transformation
\[(B.4) \quad T = \left[ \begin{array}{cc}
I & -P_{12} P_{22}^{-1} \\
0 & I \\
\end{array} \right],\]
in (B.2) while noting that \(\hat{A}_{13}\) asymptotically stable implies that \(P_{22} > 0.\) Note also that \(\hat{A}_{13}, \hat{A}_{23}, \hat{C}_3, \hat{B}_{11},\) and \(\hat{B}_{11}\) are redefined as the result of this transformation.

(ii) \(P_{22} = I,\) i.e., the realization \((\hat{A}_{11}, \hat{B}_{11})\) is input balanced.

Next, consider the all-pass equations corresponding to
\[(B.5) \quad \sigma^{-1} \left[ \begin{array}{ccc}
\hat{A}_{11} & \hat{A}_{13} & \hat{B}_{11} \\
0 & \hat{A}_{13} & \hat{B}_{21} \\
\sigma^{-1} \hat{C}_1 & \hat{C}_{23} & 0 \\
\end{array} \right] \ast \left[ \begin{array}{ccc}
\hat{A}_{33} & \hat{B}_{31} & 0 \\
\sigma^{-1} \hat{C}_2 & I & 0 \\
\sigma^{-1} \hat{C}_2 & I & 0 \\
\end{array} \right],\]
These may be written out in full as

\[
\begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} \\
0 & \tilde{A}_{33}
\end{bmatrix}
\begin{bmatrix}
P_{11} & 0 \\
0 & I
\end{bmatrix}
+ \begin{bmatrix}
P_{11} & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} \\
0 & \tilde{A}_{33}
\end{bmatrix}^* \\
\begin{bmatrix}
\tilde{B}_{11} & \tilde{B}_{12} \\
0 & \tilde{B}_{31}
\end{bmatrix}
\begin{bmatrix}
\tilde{B}_{11} & \tilde{B}_{12} \\
0 & \tilde{B}_{31}
\end{bmatrix}^* = 0
\]

(B.6)

and

\[
\begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{B}_{11} & \tilde{B}_{12} \\
0 & \tilde{B}_{31}
\end{bmatrix}
+ \begin{bmatrix}
\sigma^{-1} \tilde{C}_1 & \sigma^{-1} \tilde{C}_2 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
P_{11} & 0 \\
0 & I
\end{bmatrix} = 0,
\]

from which we get

(B.8) \[ (\tilde{B}_{31})^* = -\sigma^{-1} \tilde{C}_{23} \]

and

(B.9) \[ \tilde{A}_{12} = -\sigma^{-1} \tilde{B}_{11} \tilde{C}_{23}. \]

(B.8) and (B.9) may now be used to establish the required decomposition, while (B.8) and the (2, 2) block of (B.6) show that \( \mathcal{A}(s) \) is inner. \( \square \)

REFERENCES


